

# The pion quasiparticle in the low-temperature phase of QCD

Bastian B. Brandt<sup>1</sup>, Anthony Francis<sup>2</sup>, Harvey B. Meyer<sup>3,4</sup> and Daniel Robaina<sup>3</sup>

<sup>1</sup>*Institut für theoretische Physik, Universität Regensburg, D-93040 Regensburg, Germany*

<sup>2</sup>*Department of Physics & Astronomy, York University,  
4700 Keele St, Toronto, ON M3J 1P3, Canada*

<sup>3</sup>*PRISMA Cluster of Excellence & Institut für Kernphysik,  
Johannes Gutenberg-Universität Mainz, D-55099 Mainz, Germany*

<sup>4</sup>*Helmholtz Institut Mainz, Johannes Gutenberg-Universität Mainz, D-55099 Mainz, Germany*

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We investigate the properties of the pion quasiparticle in the low-temperature phase of two-flavor QCD on the lattice with support from chiral effective theory. We find that the pion quasiparticle mass is significantly reduced compared to its value in the vacuum, by contrast with the static screening mass, which increases with temperature. By a simple argument, near the chiral limit the two masses are expected to determine the quasiparticle dispersion relation. Analyzing two-point functions of the axial charge density at non-vanishing spatial momentum, we find that the predicted dispersion relation and the residue of the pion pole are simultaneously consistent with the lattice data at low momentum. The test, based on fits to the correlation functions, is confirmed by a second analysis using the Backus-Gilbert method.

## I. INTRODUCTION

Identifying the spectrum of excitations of strongly interacting matter at finite temperature is of central importance to understanding the nature of the medium. These excitations are encoded as poles in thermal correlation functions. For some quantum numbers, the existence of weakly interacting probes coupling to local operators make it possible to measure the properties of these excitations experimentally, at least in principle. The prime example of such a probe is the photon. In practice, a medium is created in heavy-ion collisions which appears to reach thermal equilibrium locally, the temperature decreasing with time. Therefore a weighted average of thermal photon or dilepton spectra over the spacetime history of the ‘fireball’ is obtained (see e.g. [1, 2] and Refs. therein).

In the low-temperature phase, it is natural to ask how close the properties of the excitations are to those of the known hadrons at zero temperature. Viewed globally, the spectrum does not appear to change much until temperatures close to the transition temperature are reached, where the rapid crossover to a deconfined and chirally symmetric phase takes place. This conclusion is based on the success of the hadron resonance gas model in describing equilibrium properties of the medium (particularly the equation of state and quark number susceptibilities) computed in lattice QCD [3–5], and on its success in describing particle yields in heavy-ion collisions [6, 7]. However, reliable information about individual excitations is sparse.

Here we extend our study [8] of the pion at finite temperature in two-flavor lattice QCD with support from a thermal chiral effective theory [9–11]. In [8] we performed a temperature-scan on  $16 \times 32^3$  ensembles; here we focus on one temperature ( $T = 170$  MeV) on a fine lattice ( $24 \times 64^3$ ) with high statistics, for which we also have a reference zero-temperature ensemble at the same bare parameters. Our study shows that the zero-temperature pion mass ‘splits’ at finite temperature

into a lower pion quasiparticle mass and a higher pion screening mass,

$$\begin{array}{lll} T = 0 : & \text{pion mass} = 267(2) \text{ MeV} \\ & \swarrow \quad \searrow \\ T = 169 \text{ MeV} : & \text{quasiparticle mass} = 223(4) \text{ MeV} & \text{screening mass} = 303(4) \text{ MeV}. \end{array}$$

While the quasiparticle mass is the real-part of a pole of the retarded correlator  $G_R(\omega, |\mathbf{p}| = 0)$  of the pseudoscalar density in the frequency variable, the (static) screening mass is a pole of  $G_R(\omega = 0, \mathbf{p})$  in the spatial momentum  $|\mathbf{p}|$  and represents an inverse spatial correlation length. The pion quasiparticle mass can be extracted model-independently near the chiral limit due to the dominance of its contribution to the two-point function of the axial charge density and of the pseudoscalar density. By contrast with the mass, we find that the decay constant associated with the pion quasiparticle practically retains its zero-temperature value.

By a simple argument, the dispersion relation of the pion at low momenta is given by a single parameter  $u$  (see Eq. (11) below), which in the chiral limit corresponds to the group velocity of the excitation. As in [8], we determine this parameter as the ratio of the quasiparticle mass to the screening mass,  $u \approx 0.74$ . As a new aspect, we then test whether the so-determined parameter  $u$  correctly predicts the momentum-dependence of the pion energy by looking at the two-point function of the axial charge density at non-vanishing spatial momentum. An important observation is that the chiral Ward identities also predict the residue of the pion pole in the axial-current two-point functions. Due to the difficulty of extracting real-time information from Euclidean correlation functions, testing simultaneously the predictions for the pole and the residue proves to be essential to improve the discriminative power of the analysis. Since the chiral predictions are only expected to be valid at sufficiently small momenta, we also provide an estimate of the range of validity of the effective theory.

Our analysis method of lattice correlation functions is based on fits, where the ansatz is motivated by the chiral effective theory at small frequencies and on perturbation theory at high frequency. We also present an alternative analysis, which starts by generating model-independently a locally averaged spectral function by following the Backus-Gilbert inversion method [12–17]. In a second step, a pion pole contribution with the predicted dispersion relation is assumed, allowing us to obtain an estimate for the residue. The advantage of this alternative analysis is that we do not have to formulate an explicit ansatz for the spectral density of the non-pion contributions. This point is particularly relevant since at finite spatial momentum, axial-vector excitations do contribute to the two-point function of the axial charge density.

The paper is organized as follows. Section II contains an overview of the theory expectations concerning the two-point functions of the axial current at finite temperature. Sections III and IV present the lattice QCD calculation, and our conclusions are given in section V. In appendix A, the tensor structure of the axial-current two-point functions at finite temperature is given; in appendix B, we derive the contribution of the pion to the four independent tensor structures, thus determining all the relevant residues. Finally, appendix C contains a table with the lattice correlator data.

## II. THEORY BACKGROUND

We work in the Euclidean path integral formalism, and our notation and conventions follow those used in [8]. The vector and axial-vector vectors and the pseudoscalar density are given by

$$V_\mu^a(x) = \bar{\psi}(x)\gamma_\mu \frac{\tau^a}{2}\psi(x), \quad A_\mu^a(x) = \bar{\psi}(x)\gamma_\mu\gamma_5 \frac{\tau^a}{2}\psi(x), \quad P^a(x) = \bar{\psi}(x)\gamma_5 \frac{\tau^a}{2}\psi(x), \quad (1)$$

where  $\psi$  is the isospin-doublet quark field. In appendix A, we provide a decomposition in momentum space of the Lorentz structure of the two-point functions of the axial current. For a general momentum  $p$ , they are entirely described by four ‘form factors’, which in the rest frame of the thermal medium are functions of  $p_0$  and  $\mathbf{p}^2$ . At zero-temperature, the four functions reduce to two functions of  $p^2$ , one longitudinal and one transverse. The partially-conserved axial current (PCAC) relation relates the two-point function  $\langle P^a(x)P^b(0) \rangle$  of the pseudoscalar density, as well as the  $\langle A_\mu^a(x)P^b(0) \rangle$  correlation functions to the aforementioned form factors.

In this work, we investigate the following static screening correlators,

$$\delta^{ab} G_A^s(x_3, T) = \int dx_0 d^2x_\perp \left\langle A_3^a(x)A_3^b(0) \right\rangle, \quad (2)$$

$$\delta^{ab} G_P^s(x_3, T) = \int dx_0 d^2x_\perp \left\langle P^a(x)P^b(0) \right\rangle, \quad (3)$$

where  $x_\perp = (x_1, x_2)$ . Time-dependent correlators with a general spatial momentum  $\mathbf{p}$  will also play a crucial role,

$$\delta^{ab} G_A(x_0, T, \mathbf{p}) = \int d^3x e^{i\mathbf{p}\cdot\mathbf{x}} \left\langle A_0^a(x)A_0^b(0) \right\rangle, \quad (4)$$

$$\delta^{ab} G_P(x_0, T, \mathbf{p}) = \int d^3x e^{i\mathbf{p}\cdot\mathbf{x}} \left\langle P^a(x)P^b(0) \right\rangle. \quad (5)$$

They are related by Fourier transformations to the form factors defined in appendix A, for instance

$$G_A^s(x_3, T) = \int \frac{dp_3}{2\pi} e^{-ip_3 x_3} \Pi^{L,l}(0, p_3^2). \quad (6)$$

The correlators  $G_A(x_0, T, \mathbf{0})$  and  $G_A^s(x_3, T)$  are only sensitive to the longitudinal form factor  $\Pi^{L,l}$ ; these were the cases considered in [8]. At non-vanishing momentum however, the correlator  $G_A(x_0, T, \mathbf{p})$  is sensitive to three independent form factors  $\Pi^{T,l}$ ,  $\Pi^M$  and  $\Pi^{L,l}$ .

At long distances, the screening correlator  $G_A^s(x_3, T)$  is given by

$$G_A^s(x_3, T) \stackrel{|x_3| \rightarrow \infty}{=} \frac{1}{2} f_\pi^2 m_\pi e^{-m_\pi |x_3|}, \quad (7)$$

which defines the screening pion mass  $m_\pi$  and the associated decay constant<sup>1</sup>  $f_\pi$ . The Gell-Mann–Oakes–Renner relation

$$f_\pi^2 m_\pi^2 = -m \langle \bar{\psi} \psi \rangle \quad (8)$$

holds to leading order in the chiral expansion. From Eqs. (6) and (7), the low-momentum analytic structure of the longitudinal form factor  $\Pi^{L,l}$  reads

$$\Pi^{L,l}(0, \mathbf{p}^2) = \frac{f_\pi^2 m_\pi^2}{\mathbf{p}^2 + m_\pi^2}, \quad \mathbf{p} \rightarrow 0. \quad (9)$$

More generally, expanding the denominator in the frequency,

$$\mathbf{p}^2 + m_\pi^2 \rightarrow \mathbf{p}^2 + m_\pi^2 + \frac{1}{u^2} \omega_n^2 + \dots, \quad (10)$$

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<sup>1</sup> The normalization convention is such that at zero temperature  $f_\pi \approx 92$  MeV.

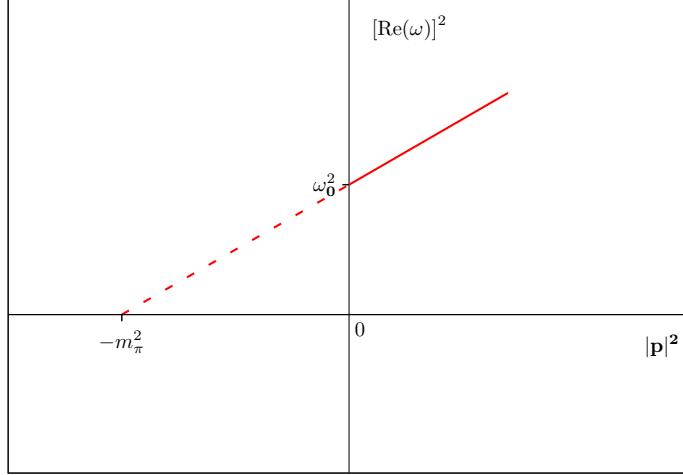


FIG. 1. Trajectory in  $(\omega, \mathbf{p})$ -plane of the pion pole in the pseudoscalar retarded correlator. At negative  $\mathbf{p}^2$  the pole corresponds to the pion screening mass, at positive  $\mathbf{p}^2$  it corresponds to the pion quasiparticle. The slope is the value of  $u^2(T)$ .

it follows that a quasiparticle (pole in the retarded correlator as a function of frequency) with the dispersion relation [10, 11]

$$\omega_{\mathbf{p}} = u(T) \sqrt{m_{\pi}^2 + \mathbf{p}^2} \quad (11)$$

exists at low momenta<sup>2</sup>. The remarkable aspect is that the parameter  $u$  determines both the (real part of the) dispersion relation of the quasiparticle and the ratio of the quasiparticle mass to the screening mass. A graphical interpretation of the dual role of the parameter  $u$  is given in Fig. 1. Here the trajectory in the frequency-momentum plane of a pole in the retarded correlator of the pseudoscalar density<sup>3</sup> corresponds to a static screening state at  $\mathbf{p}^2 = -m_{\pi}^2$ , and to a real-time quasiparticle at small positive  $\mathbf{p}^2$ . In [8], the parameter  $u$  was determined using lattice correlation functions at vanishing spatial momentum via the two estimators

$$u_m = \left[ -\frac{4m_q^2}{m_{\pi}^2} \frac{G_P(x_0, T, 0)}{G_A(x_0, T, 0)} \Big|_{x_0=\beta/2} \right]^{1/2} \quad (12)$$

$$u_f = \frac{f_{\pi}^2 m_{\pi}}{2G_A(\beta/2, T, 0) \sinh(u_f m_{\pi} \beta/2)}. \quad (13)$$

In doing so, the parametric dominance of the pion in the time-dependent Euclidean correlator at small quark masses is exploited. Good agreement was found between  $u_f$  and  $u_m$  at  $T \simeq 150$  MeV for a zero-temperature pion mass of 305 MeV. Any departure of  $u$  from unity clearly represents a breaking of Lorentz invariance due to thermal effects. In this work, one of our goals is to test whether the parameter  $u$  determined from the ratio of the quasiparticle to the screening mass, as in [18], really does predict the dispersion relation of the quasiparticle, as in Eq. (11). In order to carry out this goal, we perform an analysis of the time-dependent Euclidean correlator  $G_A(x_0, T, \mathbf{p})$  in terms of the spectral function  $\rho^A$ . They are related as follows,

$$G_A(x_0, T, \mathbf{p}) = \int_0^\infty d\omega \rho^A(\omega, \mathbf{p}) \frac{\cosh(\omega(\beta/2 - x_0))}{\sinh(\omega\beta/2)}. \quad (14)$$

<sup>2</sup> In this argument, the imaginary part of the frequency-pole is neglected. A more sophisticated argument is required to show that the damping rate of the pion quasiparticle is indeed parametrically subleading [11].

<sup>3</sup> Recall that the momentum-space Euclidean correlator  $G_E(\omega_n, \mathbf{p})$  is related to the retarded correlator via  $G_R(i\omega_n, \mathbf{p}) = G_E(\omega_n, \mathbf{p})$  for  $\omega_n \geq 0$  [18].

First we recall that at zero temperature, the Lorentz structure of the axial-current two-point function implies the following momentum dependence of the pion pole contribution,

$$G_A(x_0, T = 0, \mathbf{p}) \stackrel{|x_0| \rightarrow \infty}{\sim} \text{Res}(\omega_{\mathbf{p}}^0) \frac{e^{-\omega_{\mathbf{p}}^0 |x_0|}}{2\omega_{\mathbf{p}}^0} \quad (15)$$

where the residue is here given by

$$\text{Res}(\omega_{\mathbf{p}}^0) = (f_\pi^0 \omega_{\mathbf{p}}^0)^2, \quad \omega_{\mathbf{p}}^0 = (\mathbf{p}^2 + (\omega_0^0)^2)^{1/2}. \quad (16)$$

In terms of the spectral functions, this correlator corresponds to

$$\rho^A(\omega, T = 0, \mathbf{p}) = \text{Res}(\omega_{\mathbf{p}}^0) \delta(\omega^2 - (\omega_{\mathbf{p}}^0)^2) + \dots \quad (17)$$

At the other end of the spectrum, in the high frequency region, a leading-order perturbative calculation (see for instance [19]) yields

$$\rho^A(\omega, T, \mathbf{p}) = \theta(\omega^2 - 4m^2 - \mathbf{p}^2) \frac{N_c}{24\pi^2} (\mathbf{p}^2 + 6m^2), \quad \omega \rightarrow \infty. \quad (18)$$

We note that at non-zero momentum, the correlator  $G_A(x_0, T = 0, \mathbf{p})$  receives contributions from axial vector mesons via the transverse form factor (see appendix A),

$$G_A(x_0, T = 0, \mathbf{p}) = \int \frac{dp_0}{2\pi} e^{-ip_0 x_0} \left[ \frac{\mathbf{p}^2}{p_0^2 + \mathbf{p}^2} \Pi^T(p_0^2 + \mathbf{p}^2) + \frac{p_0^2}{p_0^2 + \mathbf{p}^2} \Pi^L(p_0^2 + \mathbf{p}^2) \right]. \quad (19)$$

The spectral functions associated with the form factors  $\Pi^T$  and  $\Pi^L$  are measured experimentally in  $\tau$  decays [20]. Since  $\Pi^T$  describes by itself the two-point function of spatial components of the axial current at vanishing spatial momentum, it cannot contain the pion pole. The latter is entirely contained in the form factor  $\Pi^L$ .

At finite temperature, the pion pole appears in all three form factors contributing to  $G_A(x_0, T, \mathbf{p})$ . Altogether, the pion contribution to the spectral function  $\rho^A$  is predicted to have the form  $\rho^A(\omega, T, \mathbf{p}) = \text{Res}(\omega_{\mathbf{p}}) \delta(\omega^2 - \omega_{\mathbf{p}}^2)$ , with the dispersion relation given by Eq. (11) and the residue by (see appendix B)

$$\text{Res}(\omega_{\mathbf{p}}) = f_\pi^2 (m_\pi^2 + \mathbf{p}^2). \quad (20)$$

For later use we also define the pion quasiparticle decay constant  $f_\pi^t$  via

$$\text{Res}(\omega_0) = (f_\pi^t \omega_0)^2. \quad (21)$$

The contribution to the Euclidean correlator then reads

$$G_A(x_0, T, \mathbf{p}) = \frac{\text{Res}(\omega_{\mathbf{p}})}{2\omega_{\mathbf{p}}} \frac{\cosh(\omega_{\mathbf{p}}(\beta/2 - x_0))}{\sinh(\omega_{\mathbf{p}}\beta/2)} + \dots \quad (22)$$

Whether the residue determined through fits to lattice correlation functions agrees with Eq. (20) provides a cross-check that the low-energy effective description is working.

### III. LATTICE SETUP

In this section we describe the analysis performed on a finite temperature ensemble of size  $24 \times 64^3$  with two degenerate dynamical light flavors. The short direction is interpreted as time and therefore the temperature is  $T = 1/24a = 169(3)\text{MeV}$  while the spatial extent amounts to  $L = 64a = 3.1\text{ fm}$ . The fields admit thermal boundary conditions in time and periodic boundary conditions in space. We use the Wilson plaquette action [21] and the O(a) improved Wilson fermion action with a non-perturbatively determined  $c_{\text{sw}}$  coefficient [22]. The configurations were generated using the MP-HMC algorithm [23] following the implementation described in [24] based on Lüsher's DD-HMC package [25]. In addition, we use a  $128 \times 64^3$ , effectively zero temperature ensemble that was made available to us through the CLS effort (labelled as O7 in [26]) with all bare parameters identical to our finite temperature ensemble. The pion mass takes a value of  $m_\pi = 270\text{MeV}$  [26] such that  $m_\pi L = 4.2$ . This additional zero temperature test ensemble allows us to compare thermal observables in a straightforward manner with their corresponding “effective zero-temperature” value calculated in the O7 ensemble.

#### A. PCAC mass

In order to check that our thermal ensemble indeed yields the same physical quark mass as its corresponding zero-temperature counterpart (O7), we use the definition of the quark mass based on the PCAC (partially conserved axial current) relation [27, 28]

$$m_{\text{PCAC}}(x_3) = \frac{1}{2} \frac{\int dx_0 d^2x_\perp \langle \partial_3^{\text{imp}} A_3^{\text{imp}}(x) P(0) \rangle}{\int dx_0 d^2x_\perp \langle P(x) P(0) \rangle} \quad x_\perp = (x_1, x_2) \quad (23)$$

where in the improvement process

$$A_\mu^a \rightarrow A_\mu^{\text{imp},a} = A_\mu^a + ac_A \partial_\mu^{\text{imp}} P^a. \quad (24)$$

The derivative  $\partial_\mu^{\text{imp}}$  is the improved lattice discretized version of the derivative following [29]. The non-perturbatively calculated coefficient  $c_A$  was taken from [30]. Note that since relation (23) stems from an operator identity, we are free to choose the direction of measurement. On the thermal ensemble, the spatial directions are longer; therefore, by measuring along these directions we obtain a longer plateau and thus more accurate determinations of the PCAC mass. The extraction can be carried out by performing a fit to a constant in the range where a plateau is observed. Its central value and error, given in Table I, are in very good agreement with the ones quoted in [31].

#### B. Pseudoscalar and axial-vector correlators

Our goal is to calculate the temperature-dependent coefficient  $u(T)$  that parametrizes the pion dispersion relation (11). In [8], we defined two estimators  $u_f$  and  $u_m$  that yielded consistent results up to  $T \simeq 170\text{MeV}$  for the case of two degenerate light flavors with  $\overline{m}^{\overline{\text{MS}}}(\mu = 2\text{GeV}) \sim 15\text{ MeV}$ ; at that quark mass, the crossover region is located around  $T_C \simeq 211\text{ MeV}$  [32]. In the thermal ensemble we are analyzing here, we have  $\overline{m}^{\overline{\text{MS}}}(\mu = 2\text{GeV}) = 12.8(1)\text{MeV}$  (see Table I), and therefore expect a slightly lower value of the transition temperature. Nevertheless, this should not affect the applicability of the chiral expansion around  $(T, m = 0)$  with  $T < T_C$ , as discussed in [8].

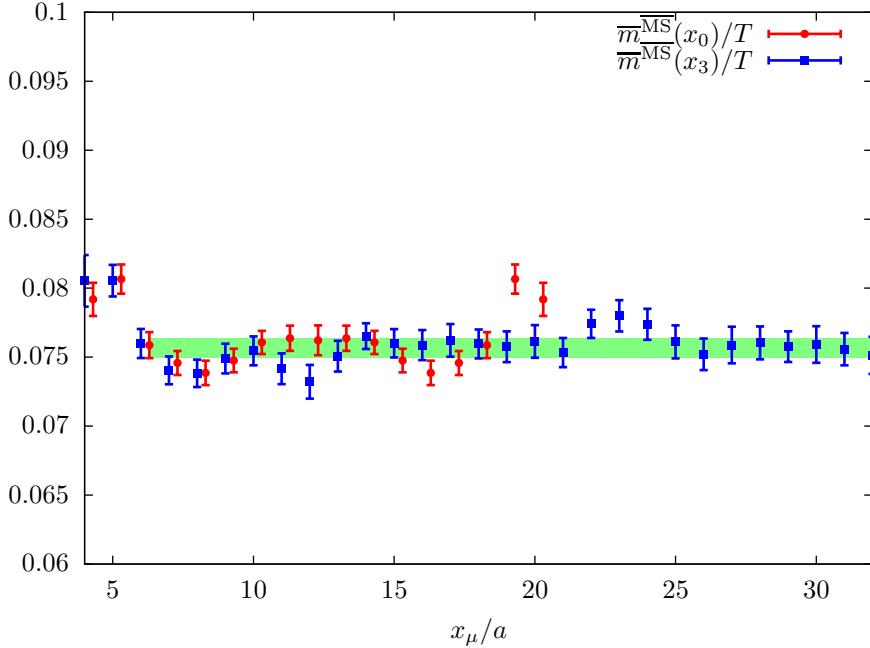


FIG. 2. The PCAC masses. The renormalization factors  $Z_A$  and  $Z_P$  are included, as well as the conversion factor from the SF to MS at a scale of  $\mu = 2\text{GeV}$ , which amounts to  $0.968(20)$  [26]. We also plot the result along the  $x_0$ -direction to show that indeed both are compatible. This can be interpreted as a check that cutoff effects are indeed small for this value of the lattice spacing.

$6/g_0^2$	5.50
$\kappa$	0.13671
$c_{\text{sw}}$	1.751496
$T_{N_\tau=24}$ [MeV]	169(3)
$T_{N_\tau=128}$ [MeV]	32(1)
$a$ [fm] [26]	0.0486(4)(5)
$Z_A$ [26]	0.793(4)
$Z_P$ [26]	0.5184(53)
$\overline{m}^{\overline{\text{MS}}}/T(\mu = 2\text{GeV})$	0.0757(7)

TABLE I. Summary of the main parameters for the  $24 \times 64^3$  finite temperature ensemble as well as for the  $128 \times 64^3$  zero temperature ensemble labelled as O7 in [26]. The quark mass is computed at and normalized with the  $T = 1/24a$  temperature. The statistics collected for two-point functions is respectively 360 and 149 configurations at  $N_\tau = 24$  and  $N_\tau = 128$ , with respectively 64 and 16 point sources per configuration, exploiting the translational invariance of the system.

We use the correlators defined in Eqs. (2–5) with the spatial momenta given by

$$\mathbf{p} = \mathbf{p}_n \equiv (0, 0, 2\pi n/L). \quad (25)$$

The improvement of the axial current was already introduced in Eq. (24). Note that since all two-point functions belong to the adjoint (or *isovector*) representation of  $SU(N_f)$  ( $N_f = 2$ ), the contributions of quark disconnected diagrams cancel out. The renormalization program is carried

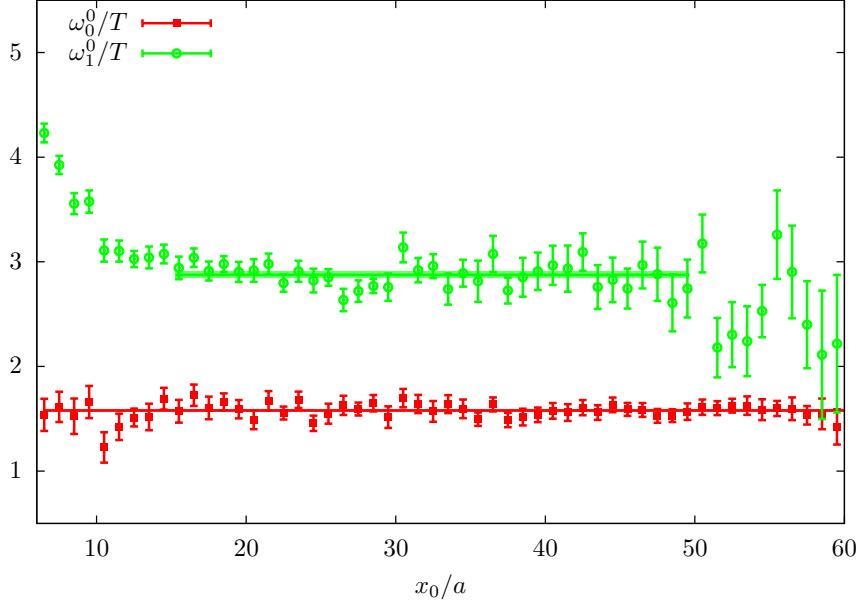


FIG. 3. Effective ‘cosh’ masses for the O7 zero-temperature ensemble in the  $\langle A_0 A_0 \rangle$  channel for  $n = 0, 1$ . The values of  $\omega_n^0 \equiv \omega_{\mathbf{p}_n}^0$  are given in units of the temperature  $T = 1/24a$  corresponding to our thermal ensemble.

out such that

$$G_A(x_0, T, \mathbf{p}) = (Z_A(g_0^2))^2 G_A(x_0, g_0^2, T, \mathbf{p}) \quad (26)$$

$$G_P(x_0, T, \mathbf{p}) = (Z_P(g_0^2))^2 G_P(x_0, g_0^2, T, \mathbf{p}) \quad (27)$$

and similarly for the screening correlators; the value of the coefficients  $Z_A$  and  $Z_P$  can be found in Table I.

#### IV. ANALYSIS OF LATTICE TWO-POINT FUNCTIONS

After the preliminary work presented in the previous section, we turn to the analysis of correlation functions in order to extract the pion properties.

##### A. The zero-temperature case

As a benchmark we analyze zero-temperature data on the O7 ensemble. Here we are able to obtain the pion energy  $\omega_{\mathbf{p}}^0$  by fitting a constant to the effective mass. The pion energy corresponding to  $\mathbf{p} = 0$  and  $\mathbf{p} = (0, 0, 2\pi/L)$  can be read off from the plot in Fig. 3. The dominance of the pion contribution, particularly in the zero-momentum case, is clearly very strong. Performing a linear fit to  $(\omega_{\mathbf{p}}^0)^2$  as a function of  $\mathbf{p}^2$ , we obtain for the slope  $u^2(T \simeq 0) = 1.01(6)$ , consistently with Lorentz invariance. The decay constant  $f_\pi^0$ , defined by Eq. (16), indeed turns out to be independent of the momentum.

Once the ground state dominates the correlator  $G_A(x_0, T \simeq 0, \mathbf{p})$ , one-state cosh fits of the form  $A_1 \cosh(\omega_{\mathbf{p}}^0(T/2 - x_0))$  with  $T = 128a$  are applied and the results are summarized in Table II. The values for  $\omega_0^0$  and  $f_\pi^0$  are in very good agreement with the ones quoted in [31].

$n$	$A_1/T^3$	$\omega_{\mathbf{p}_n}^0/T$	$\chi^2/d.o.f$	$f_\pi^0/T$	$\text{Res}(\omega_{\mathbf{p}})$
0	$8.4(3) \times 10^{-3}$	1.579(12)	0.05	0.599(8)	0.89(3)
1	$5.3(4) \times 10^{-4}$	2.88(3)	0.4	0.629(12)	3.27(15)

TABLE II. Properties of the pion at zero temperature. The index  $n$  denotes the momentum  $\mathbf{p}_n$  induced and  $\omega_{\mathbf{p}_n}^0$  corresponds to the energy of the state (in particular,  $\omega_0^0$  is the pion mass). All errors are purely statistical, and all renormalization factors are included. The fit interval begins at  $x_0/a = 6$  for the zero-momentum case and at  $x_0/a = 15$  for one unit of momentum in view of the effective mass plot of Fig. 3. Dimensionful quantities are normalized with  $T = 1/24a$ .

### B. The screening quantities $f_\pi$ and $m_\pi$ at finite temperature

A detailed description of how the extraction is carried out can be found in [8]. Here, we highlight the basic relations that lead to the extraction of the screening quantities  $f_\pi$  and  $m_\pi$  and therefore to the values of the estimators  $u_f$  and  $u_m$ .

- The screening mass  $m_\pi$  is calculated by fitting the correlation function  $G_P^s(x_3, T)$  with a two-state ansatz of the form  $\sum_{i=1}^2 A_i \cosh[m_i(L/2 - x_3)]$  with masses  $m_i$  and amplitudes  $A_i$ . The value obtained for the ground-state mass is compatible with the value obtained from the ‘cosh’ mass which is defined as the positive root of the following equation (see Fig. 4),

$$\frac{G_P^s(x_3, T)}{G_P^s(x_3 + a, T)} = \frac{\cosh[m_{\text{cosh}}(x_3 + a/2)(x_3 - L/2)]}{\cosh[m_{\text{cosh}}(x_3 + a/2)(x_3 + a - L/2)]}. \quad (28)$$

- We determine the screening pion decay constant from the correlator  $G_A^s(x_3, T)$  via Eq. (7) by applying again a two-state ‘cosh’ ansatz. The screening pion mass  $m_\pi$  also appears in  $G_A^s$ . We use this fact as a consistency check, but due to the better signal to noise ratio of the pseudoscalar channel, we quote the value extracted from  $G_P^s$  as our final result for  $m_\pi$ .

### C. Thermal time-dependent correlators at zero momentum

The estimators  $u_f$  and  $u_m$  for the pion velocity  $u$  are defined in Eqs. (12-13). Apart from the PCAC mass,  $f_\pi$  and  $m_\pi$ , they involve the time-dependent correlators  $G_A(x_0, T, \mathbf{0})$  and  $G_P(x_0, T, \mathbf{0})$ . The difference between  $u_f$  and  $u_m$  can be explained as follows. The estimator  $u_m$  is based on the dominance of the pion contribution in the correlators  $G_A$  and  $G_P$  at  $x_0 = \beta/2$ ; the estimator  $u_f$  is based on assuming that the residue is given by the screening quantities (as predicted by the chiral effective theory),  $\text{Res}(\omega_0) = f_\pi^2 m_\pi^2$ , and the dominance of the pion contribution in  $G_A$  only. The dominance in  $G_A$  is less strong an assumption than the assumption that the pion dominates  $G_P$ , since their spectral functions are related by  $\rho_P(\omega) = -\frac{\omega^2}{4m^2} \rho_A(\omega)$  at zero spatial momentum. We summarize results for  $u_f$  and  $u_m$  in Table III.

The chiral expansion around  $(T, m = 0)$  proposed in [11] assumes that one is sufficiently close to the chiral limit. In this limit, the screening pion mass  $m_\pi$  vanishes and the coefficient  $u(T)$  is indeed the velocity of a massless pion quasiparticle in the presence of a thermal bath. A deviation from unity corresponds to a violation of boost invariance. At finite but small quark mass, we showed in [8] that the consistency of  $u_f$  and  $u_m$  serves as an indicator for the applicability of the chiral expansion. Based on the results of Table III, we conclude that they are indeed consistent. The results, in particular for  $u_f$ , are in good agreement with the results obtained in [8] on ensembles

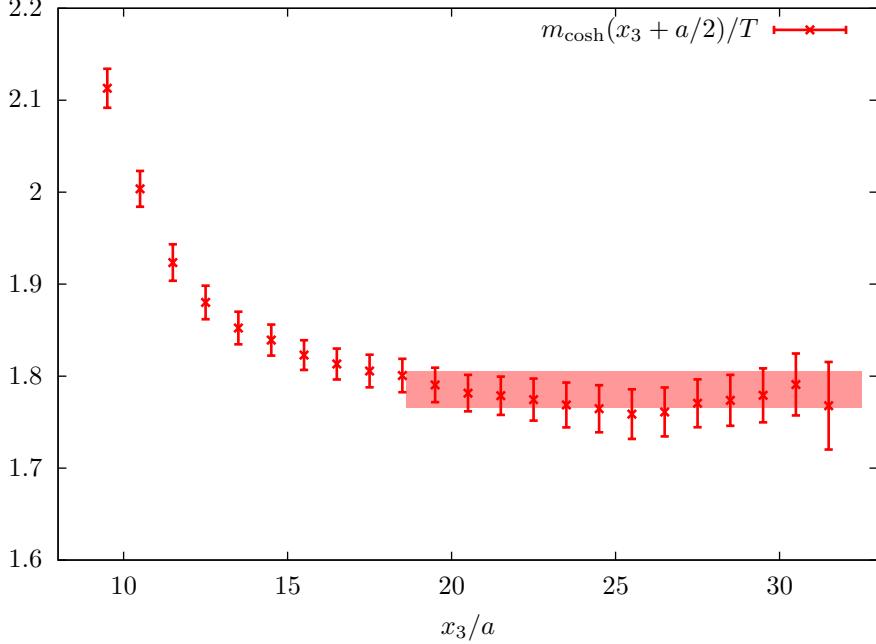


FIG. 4. Effective ‘cosh’ mass plot for the screening mass  $m_\pi$ . The plateau has been chosen to begin at  $x_3/a = 19$ .

$m_\pi/T$	1.79(2)
$f_\pi/T$	0.46(1)
$u_f$	0.76(1)
$u_m$	0.74(1)
$u_f/u_m$	1.02(1)
$\omega_0/T$	1.32(2)
$f_\pi^t/T$	0.62(1)
$\text{Res}(\omega_0)/T^4$	0.68(2)

TABLE III. Summary of the results for the  $N_\tau = 24$  thermal ensemble. All renormalization factors are included and the errors are purely statistical. The value of  $\omega_0$  is calculated using  $\omega_0 = u_m m_\pi$ . In the same way  $f_\pi^t = f_\pi/u_m$ . The value of the residue is obtained according to Eq. (20),  $\text{Res}(\omega_0) = f_\pi^2 m_\pi^2$ .

with a coarser lattice spacing and a slightly heavier quark mass. Note that to leading order,  $u$  is expected to be independent of the quark mass.

The ‘reconstructed’ correlator  $G_A^{\text{rec}}$  is defined as the thermal Euclidean correlator that would be realized if the spectral function remained the zero-temperature one. We compute it following the method first proposed in [33]. Figure 5 shows the difference between the thermal correlator and the reconstructed correlator. There is a statistically significant difference between the two correlators, which shows that a change must take place in the spectral function. Because the difference is very weakly dependent on time, the change must take place in the low-frequency part of the spectral function. We expect from the thermal chiral effective theory that the change is due to a modification of the mass and/or the residue of the pion quasiparticle. Using the numbers of Table III, the changes amount respectively to

$$\frac{\omega_0}{\omega_0^0} = 0.836(14), \quad \frac{f_\pi^t}{f_\pi^0} = 1.03(2). \quad (29)$$

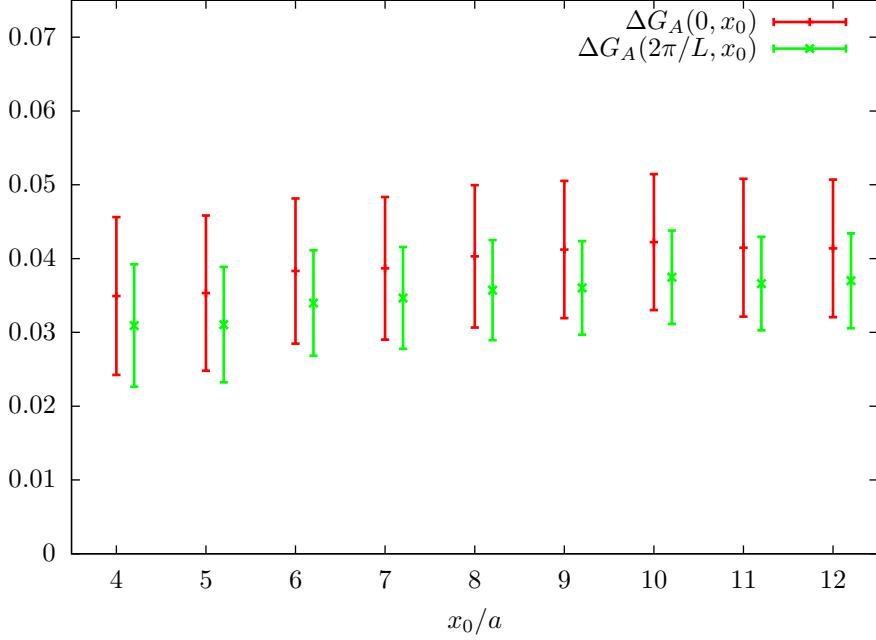


FIG. 5. The difference  $\Delta G_A(|\mathbf{p}|, x_0) \equiv [G_A(x_0, T, \mathbf{p}) - G_A^{\text{rec}}(x_0, T, \mathbf{p})]/T^3$  of the thermal correlator and the reconstructed correlator for  $|\mathbf{p}| = 0$  and  $2\pi/L$ .

We thus observe that while the pion decay constant remains unchanged at the precision level of a few percent, the pion mass decreases by 16%. Qualitatively, these results are consistent with the two-loop results in zero-temperature chiral perturbation theory given in [34, 35]. Future lattice calculations approaching the chiral limit would allow for a quantitative comparison.

#### D. The spectral function $\rho^A(\omega, p)$ at non-zero momentum

As the next step, we test the functional form of Eq. (11) at non-zero momentum. The relevant real-time pion states with energy  $\omega_{\mathbf{p}}$  have a non-zero overlap with the operator  $\int d^3x e^{i\mathbf{p}x} A_0(x)$ ; furthermore, the spectral function  $\rho_A$  becomes independent of  $\omega$  in the ultraviolet, rather than growing like  $\omega^2$ . We therefore expect to have the best sensitivity to the pion contribution in the correlator  $G_A(x_0, T, \mathbf{p})$ .

At finite temperature, the analysis of the correlator  $G_A(x_0, T, \mathbf{p})$  is more involved than at zero temperature: only at sufficiently small quark masses and momenta, and not too small  $x_0$  is the correlator parametrically dominated by the pion pole. Therefore we proceed by formulating a fit ansatz to take into account the non-pion contributions. The combination of Eqs. (17) and (18) motivates an ansatz for the spectral function reading

$$\rho^A(\omega, \mathbf{p}) = A_1(\mathbf{p}) \sinh(\omega\beta/2) \delta(\omega - \omega_{\mathbf{p}}) + A_2(\mathbf{p}) \frac{N_c}{24\pi^2} \left(1 - e^{-\omega\beta}\right) \theta(\omega - c). \quad (30)$$

The corresponding form of the correlation function then reads

$$G^A(x_0, T, \mathbf{p}) = A_1(\mathbf{p}) \cosh(\omega_{\mathbf{p}}(\beta/2 - x_0)) + A_2(\mathbf{p}) \frac{N_c}{24\pi^2} \left( \frac{e^{-cx_0}}{x_0} + \frac{e^{-c(\beta-x_0)}}{\beta - x_0} \right). \quad (31)$$

We fit  $G_A(x_0, T, \mathbf{p})$  with the ansatz given in Eq. (31) for the momenta  $\mathbf{p}_n = (0, 0, 2\pi n/L)$  with  $n = 1, 2, 3, 4, 5$ . The fit interval is chosen to be  $x_0/a \in [5, 12]$  in order to avoid cutoff effects. There

are four parameters involved,  $A_1(\mathbf{p})$ ,  $\omega_{\mathbf{p}}$ ,  $A_2(\mathbf{p})$  and  $c$ . Leaving  $\omega_{\mathbf{p}}$  as a fit parameter led to poorly constrained fits. Therefore we set the value of  $\omega_{\mathbf{p}}$  to the prediction of Eq. (11) in order to test whether the data can be described in this way. Motivated by the expected large- $\omega$  behavior of the spectral function, we quote the rescaled parameter  $\tilde{A}_2 = A_2/\mathbf{p}^2$ . Note that the quark mass is negligible compared to all the non-vanishing  $|\mathbf{p}|$  values considered here. The expected value of  $\tilde{A}_2$  is of order unity, in view of Eq. (18). Eq. (22) allows us to establish the relation between the fit parameter  $A_1(\mathbf{p})$  and the residue itself,

$$\text{Res}(\omega_{\mathbf{p}}) = 2A_1(\mathbf{p})\omega_{\mathbf{p}} \sinh(\omega_{\mathbf{p}}\beta/2). \quad (32)$$

One may further convert the result for the residue into a parameter  $b(\mathbf{p})$ , defined by

$$\text{Res}(\omega_{\mathbf{p}}) = f_\pi^2(m_\pi^2 + \mathbf{p}^2)(1 + b(\mathbf{p})). \quad (33)$$

From the chiral prediction Eq. (20), we thus expect  $b(\mathbf{p})$  to be small compared to unity if the effective description is working. The results are summarized in Table IV.

$n$	$A_1/T^3$	$\omega_{\mathbf{p}_n}/T$	$\tilde{A}_2$	$c/T$	$\text{Res}(\omega_{\mathbf{p}_n})/T^4$	$b$	$\chi^2/\text{d.o.f}$
1	$2.95(4) \times 10^{-1}$	2.19(3)	1.78(8)	6.7(3)	1.72(6)	-0.08(3)	0.06
2	$1.40(5) \times 10^{-1}$	3.73(6)	1.26(2)	6.1(1)	3.3(2)	-0.39(4)	0.15
3	$4.9(3) \times 10^{-2}$	5.40(9)	1.19(1)	7.7(1)	3.9(5)	-0.65(4)	0.35
4	$1.7(2) \times 10^{-2}$	7.1(1)	1.15(1)	9.67(9)	4.21(7)	-0.78(3)	0.49
5	$4(1) \times 10^{-3}$	8.8(1)	1.12(1)	11.7(1)	3(1)	-0.89(3)	1.04

TABLE IV. Results of fits to the axial-charge density correlator at non-vanishing momentum  $\mathbf{p}_n$ . All errors quoted are statistical, and all renormalization factors are included. The quantity  $\omega_{\mathbf{p}}/T$  is not a fit parameter; rather it is set to the value predicted by Eq. (11) with  $u(T) = u_m = 0.74(1)$ .

The fits provide a good description of the data; see the  $\chi^2/\text{d.o.f}$  values and Fig. 6. We observe that at the smallest momentum,  $|\mathbf{p}| \simeq 400$  MeV,  $b(\mathbf{p})$  really is small, pointing to a successful check of the chiral prediction. At higher momenta, the negative, order unity value of  $b(\mathbf{p})$  indicates that the residue of the pion pole is reduced. It should also be remembered that at higher momenta, neglecting the width of the quasiparticle is bound to be an increasingly poor approximation. The coefficient  $\tilde{A}_2$  is expected to be of order unity from the treelevel prediction Eq. (18). Indeed the numbers in Table IV are of order unity. One reason for the relatively large value of the coefficient at the smallest momentum could be that axial-vector excitations are contributing around the threshold  $c$ , thus adding spectral weight. The value of the threshold at  $|\mathbf{p}| \simeq 400$  MeV, is about 1.1 GeV, a value we consider to be reasonable given that the mass of the lightest axial-vector meson in nature is  $m_{a_1} \approx 1.2$  GeV.

In order to gauge the discriminative power of the test, it is interesting to ask whether a rather different model is consistent with the lattice data on  $G_A(x_0, \mathbf{p}, T)$ . We assume for this purpose that the dispersion relation and the residue have the same  $\mathbf{p}$ -dependence as at zero temperature. We therefore set  $\omega_{\mathbf{p}} = \sqrt{\omega_0^2 + \mathbf{p}^2}$ , and obtain for  $n = 1$  an equally good description of the data, with a value of the residue  $\text{Res}(\omega_{\mathbf{p}}) = 3.01(4)$  not too different from  $(f_\pi^t)^2(\omega_0^2 + \mathbf{p}^2) = 2.84(7)$ . The other fit parameters take the values  $\tilde{A}_2 = 2.42(17)$  and  $c/T = 10.2(4)$ . While the perturbative coefficient and the threshold values seem less plausible to us, we cannot completely exclude this model on the basis of the lattice data.

To summarize, we have found that the dispersion relation of the pion quasiparticle is consistent with Eq. (11), the parameter  $u$  being determined at vanishing spatial momentum. In order to test the dependence of our results on the fit ansatz made, in the next section we apply the Backus-Gilbert method.

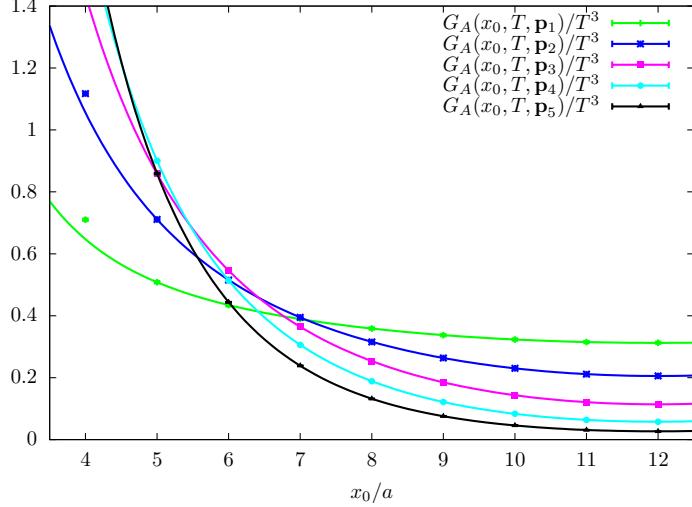


FIG. 6. Correlation functions  $G_A(x_0, T, \mathbf{p}_n)/T^3$  with  $\mathbf{p}_n = (0, 0, 2\pi n/L)$ , together with the fits resulting from the four-parameter Ansatz of Eq. (31). The corresponding parameter values are given in Table IV. All renormalization constants are included.

### E. The Backus-Gilbert method for $\rho^A(\omega, \mathbf{p})$

The Backus-Gilbert method is a method suitable for inverting integral equations like Eq. (14). It has been studied in many contexts (see e.g. [12–17]). While it has not been applied in lattice QCD, to our knowledge, the central notion of resolution function was used in [36]. We first describe the method in some generality. It is a completely model-independent approach since no ansatz needs to be made for the spectral function.

The goal is to solve the integral equation

$$G(x_i) = \int_0^\infty d\omega f(\omega) K(x_i, \omega) \quad x_i \neq 0 \quad \forall i \quad (34)$$

for the unknown function  $f(\omega)$ , given the kernel  $K(x_i, \omega)$  and given data on  $G(x_i)$ . The idea is to define an estimator  $\hat{f}(\bar{\omega})$

$$\hat{f}(\bar{\omega}) = \int_0^\infty \hat{\delta}(\bar{\omega}, \omega) f(\omega) d\omega \quad (35)$$

where  $\hat{\delta}(\bar{\omega}, \omega)$  is called the resolution function or averaging kernel. It is a smooth function concentrated around some reference value  $\bar{\omega}$ , normalized according to  $\int_0^\infty d\omega \hat{\delta}(\bar{\omega}, \omega) = 1$ , and parametrized at fixed  $\bar{\omega}$  by coefficients  $q_i(\bar{\omega})$ ,

$$\hat{\delta}(\bar{\omega}, \omega) = \sum_i q_i(\bar{\omega}) K_i(\omega), \quad (36)$$

so that  $\hat{f}$  is obtained according to

$$\hat{f}(\bar{\omega}) = \sum_{i=1}^n G(x_i) q_i(\bar{\omega}). \quad (37)$$

The goal is then to minimize the width of the resolution function. Minimizing the second moment of  $\hat{\delta}(\bar{\omega}, \omega)$  in its second argument around its first argument yields

$$q_i(\bar{\omega}) = \frac{\sum_j W_{ij}^{-1}(\bar{\omega}) R(x_j)}{\sum_{k,l} R(t_k) W_{kl}^{-1}(\bar{\omega}) R(x_l)}, \quad (38)$$

where

$$W_{ij}(\bar{\omega}) = \int_0^\infty d\omega K(x_i, \omega)(\omega - \bar{\omega})^2 K(x_j, \omega), \quad (39)$$

$$R(x_i) = \int_0^\infty K(x_i, \omega) d\omega. \quad (40)$$

We remark that  $\hat{f}(\omega)$  equals  $f(\omega)$  if the latter is constant.

The matrix  $W_{ij}(\bar{\omega})$  is very close to being singular. This is the reason why, when trying to use a data set with error bars, one needs to regulate and change the matrix  $W$  to

$$W_{ij} \rightarrow \lambda W_{ij} + (1 - \lambda) S_{ij}, \quad 0 < \lambda < 1 \quad (41)$$

where  $S_{ij}$  is the covariance matrix of the data. The value of  $\lambda$  controls the trade-off between resolution and stability. For values of  $\lambda$  close to 1, we obtain the best possible resolution. However the results tend to be unstable since the matrix is poorly conditioned and large cancellations take place among the terms in Eq. (37). Reducing  $\lambda$  improves the stability of the result at the cost of deteriorating the frequency resolution.

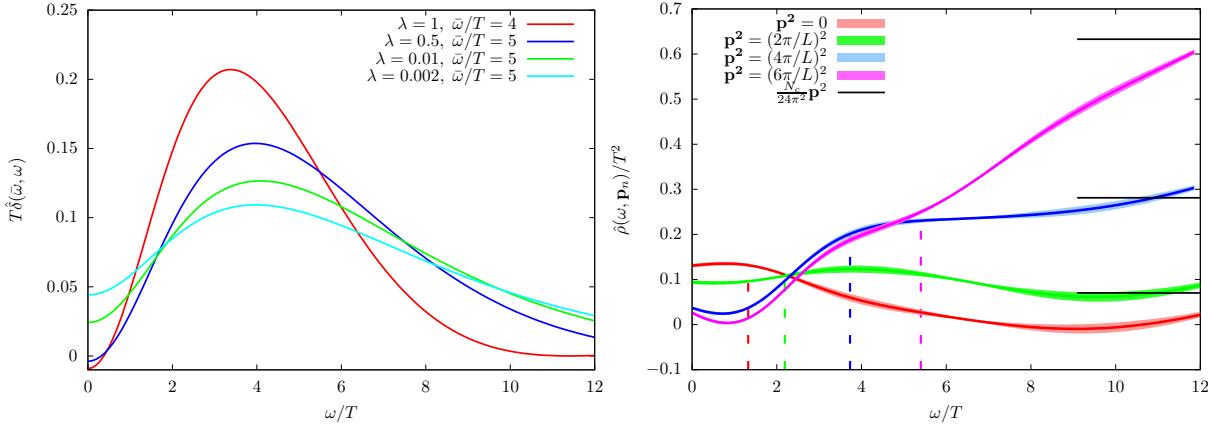


FIG. 7. Left: Some examples of resolution functions for different values of  $\lambda$  centered at  $\bar{\omega}/T$ . Right: Estimators  $\hat{\rho}(\omega, \mathbf{p}_n)/T^2$  for  $n = 0, 1, 2, 3$  together with its error shown as a band. The vertical colored dashed lines correspond to the locations of the expected positions of the poles  $\omega_{\mathbf{p}_n}$  according to Eq. (11) with  $u(T) = u_m$ . The black horizontal lines correspond to the treelevel asymptotic values of  $\rho^A(\omega, \mathbf{p})$ . All renormalization constants have been taken into account as well as the improvement program on the axial correlators. Dimensionful quantities have been made dimensionless by the appropriate power of  $T = 1/24a$ .

We apply this method to Eq. (14). In order to regularize the finite-temperature kernel at  $\omega = 0$  we rewrite the equation as

$$G_A(x_0, T, \mathbf{p}) = \int_0^\infty d\omega \left( \frac{\rho^A(\omega, \mathbf{p})}{\tanh(\omega/2)} \right) \underbrace{\left( \frac{\cosh(\omega(\beta/2 - x_0))}{\cosh(\omega\beta/2)} \right)}_{\doteq K(x_0, \omega)}. \quad (42)$$

This defines our estimator  $\hat{\rho}$  at  $\omega = \bar{\omega}$ ,

$$\hat{\rho}(\bar{\omega}, \mathbf{p}) = \int_0^\infty d\omega \hat{\delta}(\bar{\omega}, \omega) \left( \frac{\rho_A(\omega, \mathbf{p})}{\tanh(\omega\beta/2)} \right). \quad (43)$$

After regulating the problem via the covariance matrix  $S_{ij}$  as in Eq. (41), the inversion is carried out via Singular Value Decomposition. This offers the opportunity to diagnose how badly conditioned the matrix is. With all quantities made dimensionless by applying appropriate powers of the temperature, we choose  $\lambda = 2 \cdot 10^{-3}$  in the following. Typical condition numbers of the regularized matrix in Eq. (41) are  $\sim 10^8$ . The situation gets worse when  $\lambda$  approaches unity, as explained above. The results for zero momentum and the first three units of momentum are shown in the right panel of Fig. 7. As in the case of the fit, we included the points of the correlation in the interval  $x_0/a = [5, 12]$  so  $W_{ij}(\bar{\omega})$  is a  $n \times n$  symmetric matrix with  $n = 8$ . The value of  $\lambda$  was chosen equal to 0.002. This yields a relative error on  $\hat{\rho}$  of  $\sim 3 - 5\%$ , while the resolution function is displayed in the left panel of Fig. 7. One direct observation is the fact that the expected asymptotic behavior for large values of  $\omega$  is reproduced very well.

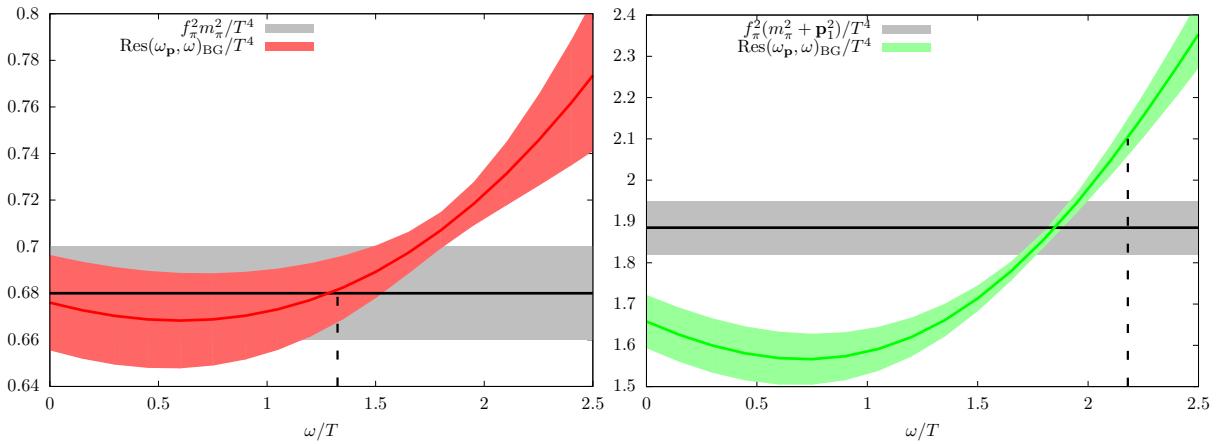


FIG. 8. The effective residue  $\text{Res}(\omega_{\mathbf{p}}, \omega)_{\text{BG}}$  as defined in Eq. (44). Left: No momentum induced,  $\mathbf{p} = 0$ . Right: One unit of momentum induced,  $\mathbf{p}_1 = (0, 0, 2\pi/L)$ . The grey band is the expectation in terms of screening quantities, Eq. (20). All renormalization factors are included. The errors arise from the statistical uncertainty. The values of  $\omega_{\mathbf{p}}$  are indicated by dashed vertical lines.

The right panel of Fig. 7 also shows the expected positions of the poles that follow from Eq. (11) as vertical colored dashed lines. We now want to test the  $\mathbf{p}$  dependence of the residue  $\text{Res}(\omega_{\mathbf{p}})$  via the following argument. If we assume that, for a given value of  $\omega$ ,  $\hat{\rho}(\omega, \mathbf{p})$  is dominated by the pion pole, we obtain the following estimator for the residue,

$$\text{Res}(\omega_{\mathbf{p}}, \omega)_{\text{BG}} = \frac{2\omega_{\mathbf{p}} \tanh(\omega_{\mathbf{p}}\beta/2)\hat{\rho}(\omega, \mathbf{p})}{\hat{\delta}(\omega, \omega_{\mathbf{p}})}. \quad (44)$$

Here we treat  $\omega_{\mathbf{p}}$  as input and calculate it using Eq. (11) with the value of  $u = u_m = 0.74(1)$  determined at zero momentum. The result as a function of  $\omega$  is shown in Fig. 8 for zero and one unit of momentum. The natural choice where  $\text{Res}(\omega_{\mathbf{p}}, \omega)_{\text{BG}}$  is expected to be the best estimator of the residue is at  $\omega \approx \omega_{\mathbf{p}}$ . Looking at Fig. 8, one sees that approximately around this value the curve intercepts the grey band, which represents the prediction Eq. (20). The latter is particularly well verified at zero momentum, while the agreement at  $|\mathbf{p}| = |\mathbf{p}_1| \approx 400\text{MeV}$  is at the ten percent level. These observations provide a further test that the pion dispersion relation Eq. (11) predicted

by the thermal chiral effective theory is consistent with the lattice two-point function of the axial charge density.

Comparing the method followed in this subsection with the previous method based on a global fit to the spectral function, the former has the advantage of not requiring an explicit parametrization of the non-pion contributions to the spectral function. This observation may be useful in other lattice studies of spectral functions.

## V. CONCLUSION

We have found that the pion quasiparticle mass is reduced significantly by thermal effects compared to its vacuum value – unlike the pion screening mass, which increases. Also, the energy cost of giving the pion quasiparticle a momentum is significantly reduced, since the ‘velocity’ is well below unity,  $u \approx 0.74$ . We have tested that the pion indeed admits a modified dispersion relation, Eq. (11), by analyzing lattice two-point functions. The test is based on requiring the consistency with the lattice data of the combined chiral prediction for the dispersion relation and the residue of the pion pole in the two-point function of the axial-charge density. These conclusions could be strengthened further by repeating the calculation at smaller quark masses and with higher statistics. Discretization errors should also be studied. Having a higher resolution in momentum could help in assessing the region of validity of the chiral effective theory.

Assuming the results hold to further scrutiny, one may wonder how much a modified mass and dispersion relation of the pion affects (a) the freeze-out mechanism in heavy-ion collisions and (b) the predictions of the hadron resonance gas model for equilibrium properties. In answering the latter question, one must take into account that the modification of the pion dispersion relation is due to the presence of hadrons in the medium, and issues of double counting arise. However, the following estimates may provide a useful first idea of the size of the effect. At the temperature of 169MeV in the two-flavor theory that we have been discussing, with a zero-temperature pion mass of 270MeV, we estimate, using the hadron resonance gas model, an isovector quark number susceptibility<sup>4</sup> amounting to  $\chi_s/T^2 = 0.42$ . In the HRG model, the pion contributes<sup>5</sup>  $\chi_s/T^2|_{\text{pions}} = 4\beta^3 \int \frac{d^3 p}{(2\pi)^3} f_B(p)(1 + f_B(p)) = 0.28$ . If the spatial-momentum integral in the pion contribution is cut off at  $p_{\max} = 400$  MeV (roughly the range of validity of the chiral effective theory that we found), the contribution is reduced to 0.11. If we instead use the modified dispersion relation with the lower quasiparticle mass  $\omega_0 = 223$  MeV and  $u = 0.74$ , the contribution for  $p < p_{\max}$  amounts again to 0.28. It is unclear whether one should include a contribution from higher momenta, given that the thermal width of the pion may then not be negligible. The numbers above illustrate that the contribution of the pion to the quark number susceptibility might not be as strongly affected as one may at first think. However, the contribution comes from softer pions, which implies a reduced amplitude of the transport peak in the two-point function of the vector current  $V_i^a(x)$ , an effect that can be tested in lattice simulations.

Determining the dispersion relation of a non-Goldstone hadron would be interesting to see whether the relatively strong change we have seen in the pion properties is specific to chiral dynamics. In general, a kinetic theory description allows one to use as primary degrees of freedom the quasiparticles specific to the temperature of interest. It is therefore much broader in applicability than the hadron resonance gas model, but requires input information on the quasiparticles. The channel treated here illustrates the importance of having guidance from an effective theory in reconstructing the gross features of the spectral function. In doing so, applying the Backus-Gilbert

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<sup>4</sup> The current is here normalized as  $\sqrt{2}V_\mu^a(x)$ , see Eq. (1).

<sup>5</sup> Here  $f_B(p) = (e^{\beta\omega_p} - 1)^{-1}$  is the Bose distribution.

method in a first step can be useful in narrowing down the region of frequency where a specific ansatz for the spectral function must be made.

## ACKNOWLEDGMENTS

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## Appendix A: Tensor structure of the axial current two-point functions

We work in the Euclidean field theory and define the correlation function in momentum space as

$$\delta^{ab} \Pi_{\mu\nu}^A(\hat{\epsilon}, k) \equiv \int d^4x e^{ik\cdot x} \langle A_\mu^a(x) A_\nu^b(0) \rangle_{\hat{\epsilon}}. \quad (\text{A1})$$

The unit vector  $\hat{\epsilon}$  points in the direction that defines the thermal boundary condition. It is  $\hat{\epsilon} = (1, 0, 0, 0)$  in the rest frame of the thermal system. By doing the change of integration variables  $x \rightarrow -x$  and using translation invariance

$$\langle A_\mu(-x) A_\nu(0) \rangle_{\hat{\epsilon}} = \langle A_\nu(x) A_\mu(0) \rangle_{\hat{\epsilon}} \quad (\text{A2})$$

and symmetry under the O(4) rotation  $x \rightarrow -x$ ,

$$\langle A_\mu(-x) A_\nu(0) \rangle_{\hat{\epsilon}} = \langle A_\mu(x) A_\nu(0) \rangle_{-\hat{\epsilon}}, \quad (\text{A3})$$

we have the symmetries

$$\Pi_{\mu\nu}^A(\hat{\epsilon}, k) = \Pi_{\nu\mu}^A(\hat{\epsilon}, -k) = \Pi_{\mu\nu}^A(-\hat{\epsilon}, -k). \quad (\text{A4})$$

To write down the tensor decomposition, we have the building blocks  $\delta_{\mu\nu}$ ,  $k_\mu$  and  $\hat{\epsilon}_\mu$  at our disposal. We can write down four structures that respect the symmetries (A4),

$$\delta_{\mu\nu}, \quad \frac{k_\mu k_\nu}{k^2}, \quad \frac{\hat{\epsilon} \cdot k}{k^2}(\hat{\epsilon}_\mu k_\nu + k_\mu \hat{\epsilon}_\nu), \quad \hat{\epsilon}_\mu \hat{\epsilon}_\nu. \quad (\text{A5})$$

We can form one projector to the subspace orthogonal to both  $\hat{\epsilon}$  and  $k$ ,

$$C_{\mu\nu}^{\text{T,t}} = \delta_{\mu\nu} - \frac{1}{1 - (\hat{\epsilon} \cdot k)^2/k^2} \left( \hat{\epsilon}_\mu \hat{\epsilon}_\nu + \frac{k_\mu k_\nu}{k^2} - \frac{1}{k^2} (\hat{\epsilon} \cdot k)(\hat{\epsilon}_\mu k_\nu + k_\mu \hat{\epsilon}_\nu) \right), \quad (\text{A6})$$

and one projector onto the component of  $\hat{\epsilon}$  which is orthogonal to  $k$ ,

$$C_{\mu\nu}^{\text{T,l}} = \delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} - C_{\mu\nu}^{\text{T,t}} = \frac{1}{1 - (\hat{\epsilon} \cdot k)^2/k^2} \left( \hat{\epsilon}_\mu - \frac{(\hat{\epsilon} \cdot k)k_\mu}{k^2} \right) \left( \hat{\epsilon}_\nu - \frac{(\hat{\epsilon} \cdot k)k_\nu}{k^2} \right). \quad (\text{A7})$$

Two possible non-transverse combinations are

$$C_{\mu\nu}^{L,l} = \frac{k_\mu k_\nu}{k^2}, \quad C_{\mu\nu}^M = \frac{1}{1 - (\hat{\epsilon} \cdot k)^2/k^2} \left( \hat{\epsilon}_\mu \hat{\epsilon}_\nu - (\hat{\epsilon} \cdot k)^2 \frac{k_\mu k_\nu}{(k^2)^2} \right). \quad (\text{A8})$$

The first one is the projector onto the direction of  $k_\mu$ . The second tensor, while not a projector, has the properties

$$C_{\mu\mu}^M = 1, \quad C_{\mu\alpha}^{T,t} C_{\alpha\nu}^M = C_{\mu\alpha}^{L,l} C_{\alpha\nu}^M = 0. \quad (\text{A9})$$

In summary, we can write

$$\Pi_{\mu\nu}^A(\hat{\epsilon}, k) = C_{\mu\nu}^{T,t} \Pi^{T,t} + C_{\mu\nu}^{T,l} \Pi^{T,l} + C_{\mu\nu}^{L,l} \Pi^{L,l} + C_{\mu\nu}^M \Pi^M. \quad (\text{A10})$$

The argument of the  $C$ 's is  $(\hat{\epsilon}, k)$ , while the argument of the form factors  $\Pi$  is  $(\hat{\epsilon} \cdot k, k^2)$ .

It is helpful to be able to invert the relation Eq. (A10) in order to project out the form factors individually. We find

$$\Pi^{L,l} = \frac{k_\mu k_\nu}{k^2} \Pi_{\mu\nu}^A, \quad (\text{A11})$$

$$\Pi^M = \frac{1}{\hat{\epsilon} \cdot k} k_\mu \Pi_{\mu\nu}^A (\hat{\epsilon}_\nu - \frac{\hat{\epsilon} \cdot k}{k^2} k_\nu), \quad (\text{A12})$$

$$\Pi^{T,l} = \frac{1}{1 - (\hat{\epsilon} \cdot k)^2/k^2} \left[ \hat{\epsilon}_\mu \Pi_{\mu\nu}^A \hat{\epsilon}_\nu - \frac{(\hat{\epsilon} \cdot k)^2}{(k^2)} \Pi^{L,l} - (1 + (\hat{\epsilon} \cdot k)^2/k^2) \Pi^M \right], \quad (\text{A13})$$

$$\Pi^{T,t} = \frac{1}{2} \left\{ \Pi_{\mu\mu}^A - \left[ \Pi^{T,l} + \Pi^M + \Pi^{L,l} \right] \right\}. \quad (\text{A14})$$

## 1. Special kinematics

When  $(\hat{\epsilon} \cdot k)^2 = k^2$ , corresponding to vanishing spatial momentum in the rest frame of the thermal system, the projectors  $C_{\mu\nu}^{T,t}$  and  $C_{\mu\nu}^{T,l}$  as well as  $C_{\mu\nu}^M$  become singular. Therefore we will define the value of the form factors in this limit by continuity. When  $\hat{\epsilon}$  and  $k$  are collinear, there are only two independent tensor structures,

$$\Pi_{\mu\nu}^{A,\text{col}}(\hat{\epsilon}, k) = \left( \delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \hat{\Pi}_T(k^2) + \frac{k_\mu k_\nu}{k^2} \hat{\Pi}_L(k^2). \quad (\text{A15})$$

Applying the relevant projectors as in Eqs. (A11–A14), one finds that

$$\Pi^{T,t} = \Pi^{T,l} = \hat{\Pi}_T, \quad \Pi^{L,l} = \hat{\Pi}_L, \quad \Pi^M = 0. \quad (\text{A16})$$

When  $\hat{\epsilon} \cdot k = 0$ , corresponding to the static correlators,  $C_{\mu\nu}^{T,l}$  becomes equal to  $C_{\mu\nu}^M$ . Therefore, in that situation the Euclidean correlator is only sensitive to the sum of the two corresponding form factors,  $(\Pi^M + \Pi^{T,l})$ . Eq. (A12) nonetheless provides an unambiguous definition of  $\Pi^M$  if, expressed in the rest frame,  $\lim_{k_0 \rightarrow 0} \Pi_{0i}^A/k_0$  is known. The latter limit, however, requires an analytic continuation of the Euclidean correlator.

## 2. The zero-temperature limit

At zero temperature, it is natural to parametrize the correlation function as

$$\Pi_{\mu\nu}^A(k) = \left( \delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} \right) \Pi^T(k^2) + \frac{k_\mu k_\nu}{k^2} \Pi^L(k^2). \quad (\text{A17})$$

Applying the same projectors as in Eqs. (A11–A14) onto the correlation function (A17), and requiring that the same result be obtained in the zero-temperature limit, we obtain

$$\Pi^{L,l}(\hat{\epsilon} \cdot k, k^2) \rightarrow \Pi^L(k^2), \quad (A18)$$

$$\Pi^M(\hat{\epsilon} \cdot k, k^2) \rightarrow 0, \quad (A19)$$

$$\Pi^{T,l}(\hat{\epsilon} \cdot k, k^2) \rightarrow \Pi^T(k^2), \quad (A20)$$

$$\Pi^{T,t}(\hat{\epsilon} \cdot k, k^2) \rightarrow \Pi^T(k^2). \quad (A21)$$

### 3. Relation to the correlators of the pseudoscalar density

We define

$$\delta^{ab} \mathcal{A}_\mu(\hat{\epsilon}, k) = \int d^4x e^{ikx} \langle A_\mu^a(x) P^b(0) \rangle, \quad (A22)$$

$$\delta^{ab} \mathcal{P}(\hat{\epsilon}, k) = \int d^4x e^{ikx} \langle P^a(x) P^b(0) \rangle. \quad (A23)$$

We note the symmetry relations

$$\mathcal{A}_\mu(\hat{\epsilon}, k) = -\mathcal{A}_\mu(-\hat{\epsilon}, -k), \quad (A24)$$

$$\int d^4x e^{ikx} \langle P(x) A_\nu(0) \rangle_{\hat{\epsilon}} = \mathcal{A}_\nu(\hat{\epsilon}, -k), \quad (A25)$$

respectively from O(4) invariance and from translation invariance.

In [8], Eq. (A7) and (A8), taking into account Eq. (A25) and Eq. (A4), the Ward identities

$$2m \mathcal{A}_\mu(\hat{\epsilon}, k) = ik_\alpha \Pi_{\mu\alpha}^A(\hat{\epsilon}, k), \quad (A26)$$

$$4m^2 \mathcal{P}(\hat{\epsilon}, k) = k_\mu \Pi_{\mu\alpha}^A(\hat{\epsilon}, k) k_\alpha + m \langle \bar{\psi} \psi \rangle \quad (A27)$$

were derived. Inserting our tensor decomposition of  $\Pi_{\mu\alpha}^A(\hat{\epsilon}, k)$ , we find

$$2m \mathcal{A}_\mu(\hat{\epsilon}, k) = ik_\mu \Pi^{L,l}(\hat{\epsilon} \cdot k, k^2) + i(\hat{\epsilon} \cdot k) \frac{\hat{\epsilon}_\mu - (\hat{\epsilon} \cdot k/k^2) k_\mu}{1 - (\hat{\epsilon} \cdot k)^2/k^2} \Pi^M(\hat{\epsilon} \cdot k, k^2), \quad (A28)$$

$$4m^2 \mathcal{P}(\hat{\epsilon}, k) = k^2 \Pi^{L,l}(\hat{\epsilon} \cdot k, k^2) + m \langle \bar{\psi} \psi \rangle. \quad (A29)$$

### Appendix B: On the residue of the pion pole

In this appendix, we use the general results of the previous section in the rest frame of the thermal system,  $\hat{\epsilon}_\mu = (1, 0, 0, 0)$ . The form factors are thus functions of  $k_0$  and  $k^2$  and the dependence on  $\hat{\epsilon}$  is no longer indicated explicitly. All expressions for correlation functions in this section refer exclusively to the pion contribution.

In [8], it was shown that the residue of the pion pole in the two-point function of  $A_0$  at vanishing spatial momentum is  $\text{Res}(\omega_0) = f_\pi^2 m_\pi^2$ . In order to determine the form of the residue at finite momentum, we parametrize the residue as

$$\text{Res}(\omega_{\mathbf{k}}) = f_\pi^2(m_\pi^2 + \lambda \mathbf{k}^2). \quad (B1)$$

To determine the parameter  $\lambda$ , we will exploit the fact that the spectral representation of the two-point function of  $A_0$  in terms of real-time excitations must agree with the spectral representation

in terms of screening states. From the former point of view, the pion contribution to the correlator in momentum space takes the form

$$\Pi_{00}^A(k) = \frac{f_\pi^2(m_\pi^2 + \lambda\mathbf{k}^2)}{k_0^2 + \omega_{\mathbf{k}}^2}, \quad (\text{B2})$$

with  $\omega_{\mathbf{k}}$  given in Eq. (11). From the ‘screening’ point of view, the residue must be proportional to  $k_0^2$  at small  $k_0^2$  (here we invoke the analytic continuation in the frequency, away from the Matsubara values  $k_0 = 2\pi T n$ ). This is so because the screening pion is odd under the Euclidean time reversal  $x_0 \rightarrow -x_0$ , while  $A_0$  is even. Thus we can write

$$\Pi_{00}^A(k) = \frac{-|F|^2 k_0^2}{\mathbf{k}^2 + \frac{k_0^2}{u^2} + m_\pi^2} \quad (\text{B3})$$

for some parameter  $F$  to be determined. Equations (B2) and (B3) must agree when the numerators are evaluated at the pole,  $k_0^2 = -\omega_{\mathbf{k}}^2$ . From here we learn the following,

$$|F| = \frac{f_\pi}{u^2}, \quad \lambda = 1. \quad (\text{B4})$$

This shows in particular that the residue has the form given in Eq. (20). Essentially the same argument was already used in [8] to determine the residue of the pion pole in the two-point function of the pseudoscalar density,

$$\mathcal{P}(k) = -\frac{\langle \bar{\psi}\psi \rangle^2 u^2}{4f_\pi^2} \frac{1}{k_0^2 + \omega_{\mathbf{k}}^2}. \quad (\text{B5})$$

We note that for a one-pole contribution, factorization relations such as

$$|\mathcal{A}_0(k)|^2 = |\mathcal{P}(k)| |\Pi_{00}^A(k)| \quad (\text{B6})$$

hold. The phase of  $\mathcal{A}_0$  can then be determined through its form at vanishing spatial momentum given in [8].

### 1. The pion contribution to $\Pi_{\mu\nu}^A$

Having found the residue of the pion pole in the various two-point functions of the axial current, we give for completeness the pion contribution to the form factors defined in Eqs. (A11–A14),

$$\Pi^{L,1}(k_0, k^2) = -\frac{f_\pi^2 m_\pi^4 u^2}{k^2(k_0^2 + \omega_{\mathbf{k}}^2)}, \quad (\text{B7})$$

$$\Pi^M(k_0, k^2) = \frac{f_\pi^2 m_\pi^2 \mathbf{k}^2 (1 - u^2)}{k^2(k_0^2 + \omega_{\mathbf{k}}^2)}, \quad (\text{B8})$$

$$\Pi^{T,1}(k_0, k^2) = \frac{f_\pi^2 \mathbf{k}^2 (1 - u^2)}{k_0^2 + \omega_{\mathbf{k}}^2}, \quad (\text{B9})$$

$$\Pi^{T,t}(k_0, k^2) = 0. \quad (\text{B10})$$

The first is obtained from Eq. (A29), then the second from Eq. (A28), the third by using Eq. (A13) and the first two results. Via Eq. (A10), the form factors allow one to obtain the entire tensor  $\Pi_{\mu\nu}^A$ .

These calculations could be greatly expedited by using an effective Lagrangian, as written down in [11]. However it is also instructive to derive the results above directly within QCD.

### Appendix C: Lattice correlation functions

$x_0/a$	$n = 0$	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$
0	$2.9921(9) \times 10^3$	$2.9962(9) \times 10^3$	$3.0079(9) \times 10^3$	$3.0260(9) \times 10^3$	$3.0496(9) \times 10^3$	$3.0774(9) \times 10^3$
1	$9.200(5) \times 10^1$	$9.335(5) \times 10^1$	$9.698(5) \times 10^1$	$1.0213(5) \times 10^2$	$1.0813(5) \times 10^2$	$1.1447(5) \times 10^2$
2	$5.65(1) \times 10^0$	$6.27(1) \times 10^0$	$7.83(1) \times 10^0$	$9.77(1) \times 10^0$	$1.171(1) \times 10^1$	$1.343(1) \times 10^1$
3	$1.566(6) \times 10^0$	$1.903(5) \times 10^0$	$2.681(5) \times 10^0$	$3.527(4) \times 10^0$	$4.222(4) \times 10^0$	$4.699(5) \times 10^0$
4	$5.19(5) \times 10^{-1}$	$7.10(4) \times 10^{-1}$	$1.117(3) \times 10^0$	$1.491(3) \times 10^0$	$1.721(3) \times 10^0$	$1.801(3) \times 10^0$
5	$4.02(5) \times 10^{-1}$	$5.08(4) \times 10^{-1}$	$7.11(3) \times 10^{-1}$	$8.58(2) \times 10^{-1}$	$9.00(2) \times 10^{-1}$	$8.59(2) \times 10^{-1}$
6	$3.84(5) \times 10^{-1}$	$4.34(4) \times 10^{-1}$	$5.16(3) \times 10^{-1}$	$5.46(2) \times 10^{-1}$	$5.14(1) \times 10^{-1}$	$4.44(1) \times 10^{-1}$
7	$3.77(5) \times 10^{-1}$	$3.90(4) \times 10^{-1}$	$3.95(3) \times 10^{-1}$	$3.64(2) \times 10^{-1}$	$3.06(1) \times 10^{-1}$	$2.386(7) \times 10^{-1}$
8	$3.72(5) \times 10^{-1}$	$3.59(4) \times 10^{-1}$	$3.16(2) \times 10^{-1}$	$2.54(1) \times 10^{-1}$	$1.887(9) \times 10^{-1}$	$1.319(6) \times 10^{-1}$
9	$3.68(5) \times 10^{-1}$	$3.37(4) \times 10^{-1}$	$2.64(2) \times 10^{-1}$	$1.85(1) \times 10^{-1}$	$1.217(8) \times 10^{-1}$	$7.55(5) \times 10^{-2}$
10	$3.66(5) \times 10^{-1}$	$3.23(4) \times 10^{-1}$	$2.30(2) \times 10^{-1}$	$1.44(1) \times 10^{-1}$	$8.35(8) \times 10^{-2}$	$4.59(5) \times 10^{-2}$
11	$3.65(5) \times 10^{-1}$	$3.15(4) \times 10^{-1}$	$2.11(2) \times 10^{-1}$	$1.21(1) \times 10^{-1}$	$6.37(7) \times 10^{-2}$	$3.11(5) \times 10^{-2}$
12	$3.65(5) \times 10^{-1}$	$3.13(4) \times 10^{-1}$	$2.06(2) \times 10^{-1}$	$1.14(1) \times 10^{-1}$	$5.77(8) \times 10^{-2}$	$2.67(6) \times 10^{-2}$

TABLE V. Renormalized correlation function  $G_A(x_0, T, \mathbf{p}_n)/T^3$ . All errors quoted are statistical.

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